Historical Perspectives on Mathematical Elegance — To What Extent is Mathematical Beauty in the Eye of the Beholder?

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This paper outlines the history of a famous school mathematics problem — originally formulated (we believe) by Isaac Newton. It appeared in a U.S. arithmetic text in 1834 and became the source of a controversy that lasted for at least 60 years. We offer two quite different solutions to the problem and provide details of the controversy that emerged. The concept of mathematical elegance, and the difficulty of specifying criteria for elegance, are discussed.

Introduction: Elegant Solutions to Mathematics Problems

Some years ago the first author of this paper, writing with Gina Del Campo, explored the concept of elegance as it applies to mathematical problem solving (see Del Campo & Clements, 1990). In the course of the discussion, the following statement was made:

> An elegant strategy has the property that it is recognised as a "very good" method by other problem solvers once they become aware of it. Elegance as we have described it implies not only a deeper than usual awareness of the structure of the problem, but also a creative ability to apply a procedure not suggested by the structure. While the construction of an "elegant" solution is a personal achievement, it is something which is readily recognised as "worthy" by others in a position to appreciate it. (p. 61)

Del Campo and Clements argued that although the sincere efforts of any problem solver should be valued, especially in an education context, "some solution strategies are more elegant (or, if you like, better) than others" (p. 61).

To support their claim, Del Campo and Clements (1990) considered the following problem:

> Imagine if you wrote the integers from 1 to 999999, one after the other (i.e., 1, 2, 3, …, 999998, 999999). How many times altogether would you have written the digit "1"? (In writing 1211, for example, you have write "1" three times.) (p. 61)

Del Campo and Clements offered the following "elegant" response to this task:

> Instead of writing 1 to 999999, I'll write 0 to 999999 as follows:
>
> 000000, 000001, 000002, … and so on, to … 999998, 999999.
>
> Now, altogether there are a million 6-digit numbers here, so there are 6 million digits in total (1000000 × 6). Each digit 0, 1, 2, 3, 4, 5, 6, 7, 8, 9 appears exactly the same number of times. Therefore: the number of "1s" is 6000000 ÷ 10 = 600000. (p. 61)

Del Campo and Clements (1990) maintained that this was "the *best* [original emphasis] possible solution", and added, "there is a certain genius evident". Del Campo and Clements then drew educational implications from their example, in the following terms. "Mathematics teachers should not aim to produce learners who will merely strive to reproduce teacher methods and 'basic facts'" (p. 61). They claimed that school students who have learned to think of mathematics as "formulae and rules" rarely generate elegant solutions to mathematics problems or elegant mathematical proofs. Most feel no need to strive for elegance because they are rewarded by giving correct solutions to standard low-level tasks.

The authors have asked more than 400 undergraduate students (at an Asian university, an American university, and an Australian university) — all of whom were training to become middle-school or secondary mathematics teachers — to attempt the "1 to 999999" task (see above). None of these students produced an elegant solution, and of the 20 percent (approximately) who managed to obtain a correct solution, almost all used a relatively lengthy, pedestrian, "listing" approach.

Mathematicians and mathematics educators have often referred to the concept of "elegance" in mathematics and have urged teachers to encourage their students to strive for elegance in mathematical problem solving and in mathematical proof (e.g., Dijkstra, 2004; Dreyfus & Eisenberg, 1986; 1990; Penrose, 1974; Poincare, 1913). Mathematicians and mathematics educators are less in agreement, however, on the criteria for deciding what constitutes an elegant, or "beautiful", solution to a problem. They accept that an elegant solution, or proof, requires "out-of-the-box thinking" but general criteria that permit effective out-of-the-box thinking to be identified and assessed reliably, across a wide range of mathematics problems, have not been specified.

Furthermore, there is evidence that although many school students recognise that some teachers value creative solutions, the students themselves are not prepared to attempt such solutions (Healy & Hoyles, 1998). Sometimes, of course, a student's preference for one type of solution, or proof, over another is related to that student's unfamiliarity with, or lack of confidence in dealing with, the non-preferred type. Thus, for example, Tall (1979) showed that first-year university students strongly preferred generic forms of proof to proofs by contradiction, even in cases where the latter was, from a mathematician's perspective, much briefer and more elegant.

The remainder of this paper is mainly concerned with attempts, by adults, to solve a mathematics task — known as the "pasturage problem" — that appeared in a school textbook in the United States of America in 1834. The task was, we believe, originally formulated by Isaac Newton. The story draws attention to the difficulty of establishing criteria for identifying elegant solutions to mathematics problems.

Tales of Oxen Eating Grass: Episodes Surrounding Attempts to Solve a Problem that Appeared in a School Arithmetic Textbook in the 1830s

In the late 1820s and early 1830s a certain Frederick Emerson wrote three school arithmetic textbooks (Emerson, 1829, 1832, and 1834) that were intended to cover all the arithmetic needed by students at all levels in schools in the United States of America. The first of these books, *The North American Arithmetic, Part First*, was designed for children aged between 5 and 8 years. It consisted of oral lessons, "the slate and pencil not being required" (Emerson, 1829, p. 2). *The North American Arithmetic, Part Second* (Emerson, 1832), was a much larger book of 192 pages, and was intended to provide "a complete system of mental and written arithmetic, sufficiently extensive for all the common purposes of business" (Emerson, 1838, p. 1). *The North American Arithmetic, Part Third* (Emerson, 1834), a still larger book of 288 pages, was designed for advanced scholars, and provided "a copious development of the higher operations, and an extensive range of commercial information" (Emerson, 1838, p. 1). It was aimed at "scholars who are to be educated for the business of the counting room, or for the duties of any public office, as well as those who are to pursue a full course of liberal education" (Emerson, 1838, p. 1). Before he wrote the books Emerson had been "Principal of the Department of Arithmetic in Boylston School, in Boston" (Emerson, 1829, p. 1)

The last section of *The North American Arithmetic, Part Third* comprised 137 miscellaneous problems, and this paper is especially concerned with Problem 137, that is to say, the very last problem in a textbook designed for advanced learners. It could be expected to be difficult, and it was. The problem is reproduced in Figure 1.

If 12 oxen eat up $3\frac{1}{2}$ acres of grass in 4 weeks, and 21 oxen eat up 10 acres in 9 weeks, how many oxen will eat up 24 acres in 18 weeks; the grass being at first equal in every acre, and growing uniformly? (From Emerson, 1834, p. 286)

Figure 1. The pasturage problem (from Emerson, 1834, p. 286).

A quick reading of this problem might suggest that its solution could be reached without much ingenuity. It has all the trappings of a standard, ratio-type problem. The reality, however, was something quite different, and this paper sketches the history of how, after the publication of *Book Third* (Emerson, 1834), the problem quickly achieved a notoriety within educated classes in the North-Eastern States of the United States.

As well as providing a brief history to events surrounding the problem, the article will also provide two solutions to the problem — one of which will be an algebraic solution that we, ourselves, developed.

Our Initial Solution (February 2006) to Emerson's Pasturage Problem

When we first attempted to solve the task our initial inclination was to try to set up appropriate ratios. However, we quickly decided that that was not helping us much. We then decided to identify, and define, the key variables that were involved as precisely as possible, and to attempt an algebraic solution. The decision to follow that path was one that might have been expected of us, given our mathematical backgrounds, but the task of actually identifying and defining the variables demanded some out-of-the-box thinking that incorporated ideas not mentioned in the statement of the problem.. We finally arrived at, and were pleased with, the following solution.

Let:

Initial (assumed to be constant across the acreage) height of grass be h";
Rate of growth of grass be g" per week;
Each ox eat d" across an acre per week.
At the end of one week, with one ox, one acre of grass is $(h + g - d)$" high.
At the end of one week, with n oxen, one acre of grass is $(h + g - nd)$" high.
If there are A acres of grass,
after one week with n oxen, the height of the grass is $(h + g - nd/A)$";
after two weeks with n oxen, the height of the grass is
$[(h + g - nd/A) + (g - nd/A)]$"
$= (h + 2g - 2nd/A)$" (n is a discrete variable)
after x weeks with n oxen, the height of the grass is $(h + xg - xnd/A)$".
So, from the given data,
$h + 4g - 4 \times 12 \times d/3.5 = 0$, that is to say
$h + 4g - 4 \times 96d/7 = 0$ (1)
and $h + 9g - 189d/10 = 0$ (2)

(2) – (1) gives $5g = 363d/70$, and hence

$g = 363d/350$ (3)

From the last set of given data, we ask

If $h + 18g - 18nd/24 = 0$, what is n? (4)

From (1), $h = 96d/7 - 4g$. Replace h by that expression in the equation in (4).

$96d/7 + 14g - 3nd/4 = 0$ (5)

From (3), $g = 363d/350$. Put that in (5).

$96d/7 + 14 \times 363d/350 - 3nd/4 = 0$. Dividing each term by d gives

$3n/4 = 96/7 + 14 \times 363/350$

$3n/4 = (4800 + 5082)/350$

which gives $n = 37\frac{113}{175}$, or 37.645.

An Outline of the Controversy Arising from Emerson's (1834) Pasturage Problem

According to Emerson (1838), in June 1835, a premium of $50 was offered, publicly, for the most "lucid analytical solution" of Question 137 in the *Third Part* of Emerson's *North American Arithmetic*. Just who it was that offered the prize is not yet known by the present authors, but we do know that subsequently a committee was appointed to examine the solutions presented. We do not know the composition of the committee, but we do it was chaired by a Mr P. Mackintosh and that subsequently it reported that of the 112 solutions submitted, only 48 had given the correct answer — despite the fact that almost all submissions had come from mathematicians or teachers of mathematics. After excluding all submissions with incorrect answers, the committee reduced the remaining number by excluding those which were "algebraical and, also, those which were performed by *position* or by *proportion*; retaining for the comparative examination, such only as were strictly analytical" (p. 110). The committee awarded the prize to a certain James Robinson, Principal of the Department of Arithmetic at Bowdoin School, in Boston.

The Award-Winning Solution to the Pasturage Problem

The following award-winning submission was reproduced from pages 69 and 70 of Emerson (1838):

> It is evident that a part of the given number of oxen, in each condition, must be supported by the grass at *first standing on* the given number of acres, and that the remaining part must be supported by the *growth*. It is also evident that the number of oxen that can be supported by the grass at first standing on the ground, must be in a direct ratio to the number of acres., and in an inverse ratio to the time of grazing. And it is further obvious, that the number of oxen that can be supported by the growth of the grass, must be in a direct ratio to the number of acres, without any regard to the time of grazing; because the number of oxen that would consume the growth of any given number of acres during any given time, would consume the same growth continually.
>
> By the first condition of the question, 12 oxen consume $3\frac{1}{2}$ acres of grass, it would require $\frac{20}{7}$ as many oxen to consume 10 acres of grass and its growth in the same time. To consume the same in 9 weeks, would require only $\frac{4}{9}$ as many oxen; and $34\frac{2}{7}$ oxen multiplied by $\frac{4}{9}$ are $15\frac{5}{21}$ oxen.
> By the second condition, 21 oxen consume 10 acres of grass and its growth in 9 weeks; – and 21 oxen less $15\frac{5}{21}$ oxen are $5\frac{16}{21}$ oxen. Then it follows that $5\frac{16}{21}$ oxen in 9 weeks would consume the growth of 10 acres of grass during the remaining 5 weeks. To consume the growth of 10 acres during 9 weeks, would require $\frac{9}{5}$ as many oxen, and $5\frac{16}{21}$ oxen multiplied by $\frac{9}{5}$ are $10\frac{13}{35}$ oxen.

Hence it is evident that $10\frac{22}{35}$ oxen, in 9 weeks, would consume the grass at first on the 10 acres;– and it is also evident that $10\frac{13}{35}$ oxen, in 9 weeks, would consume the growth of the 10 acres of grass during the 9 weeks.

The 24 acres in the third condition being $\frac{24}{10}$ or $2\frac{2}{5}$ times 10 acres, it would require $2\frac{2}{5}$ times $10\frac{22}{35}$ oxen to consume the grass at first on the 24 acres;– and $10\frac{22}{35}$ oxen multiplied by $2\frac{2}{5}$ are $25\frac{89}{175}$ oxen. To consume the same in 18 weeks would require only $\frac{9}{18}$, or $\frac{1}{2}$ as many oxen;– and $25\frac{89}{175}$ oxen divided by 2, are $12\frac{132}{175}$ oxen. And to consume the growth of the 24 acres of grass during the 18 weeks, would require $2\frac{2}{5}$ times $10\frac{13}{35}$ oxen;– and $10\frac{13}{35}$ oxen multiplied by $2\frac{2}{5}$ are $24\frac{156}{175}$ oxen.

Lastly, $12\frac{132}{175}$ oxen plus $24\frac{156}{175}$ oxen are $37\frac{113}{175}$ oxen, the number required.

Exactly why the committee rejected algebraic solutions, and how it arrived at its decision that the above solution, was the most lucid, is not known. We would agree that Robinson's solution required out-of-the-box thinking, especially in relation to structuring the given data. However, it seems to us that the solution required substantial, and in our opinion, "ugly", numerical calculations that were unnecessary in our algebraic solution.

Furthermore, a careful examination of Robinson's solution reveals that throughout there is repeated references to fractional parts of oxen. The number of oxen, n say, that would eat A acres of grass in W weeks is, presumably, a discrete variable, taking integer values only. Intuitively, one would feel that n is a function of a number of continuous variables: A, W, the initial (assumed to be constant) height of grass, the rate of growth of grass week, and the depth of grass an ox would eat across an acre per week. Mathematicians often allow an obviously discrete variable to masquerade as a continuous variable, and then round their answers off, at the end, to an integer. But that is only legitimate if the internal structure of the situation being analysed would not be compromised by the decision to allow the discrete variable to be regarded as a continuous variable. It is not clear to us whether Robinson's method was acceptable, from that perspective.

Cajori (1890) provides further details in relation to the history of the oxen-eating-grass problem. In the late 1850s, Robinson's solution was examined by the National Teachers' Association, in Washington, an din Volume 2 (number 3) of the *Mathematical Monthly*, published in 1859, the Hon. Finley Bigger, then Register of the U.S. Treasury, asserted that the problem was "susceptible of two constructions" and could have different answers depending on the interpretation. However, the editor of the *Mathematical Monthly,* at that time, rejected Bigger's contention, and after offering his own algebraic solution to the problem, for *Monthly* readers, he asserted that there was only one correct answer, specifically $37\frac{113}{175}$ oxen.

But the matter did not rest there. In 1882, Dr Artemas Martin, editor of the *Mathematical Magazine* (see Volume 1, pages 17 and 43) revisited the problem. After commenting that he did not consider Robinson's solution to be "very lucid", and that he particularly did not approve of the way Robinson's solution made "mincemeat" of the oxen, Martin published several solutions to the problem. Cajori (1890) commented:

> Neither Mr Emerson, nor the committee, nor Mr Robinson, nor Mr Bigger, nor the national Teachers' Association, nor the *Mathematical Monthly*, alludes to the fact that the question is taken from the *Arithmetica Universalis* of Sir Isaac Newton, published in 1704, which contains a "lucid analytical solution". Mr Emerson's statement of the problem differs from that of Newton in this, that, owing to a misprint the fraction $\frac{1}{2}$ instead of $\frac{1}{3}$ is given by the former in the number of acres contained in the first pasture, which mistake produces the absurd result of $37\frac{113}{175}$ oxen, instead of 36. (p. 110)

We have not been able to locate Newton's own method for solving the problem, but we used our own algebraic approach, outlined towards the beginning of this paper, to obtain the answer 36. Here is our solution to Newton's original question.

Our Solution to the Original Isaac Newton Problem

Let:

- Initial (assumed to be constant) height of grass be h";
- Rate of growth of grass be g" per week;
- Each ox eats d" across an acre per week.

At the end of one week, with one ox, one acre of grass is $(h + g - d)$" high.

At the end of one week, with n oxen, one acre of grass is $(h + g - nd)$" high.

If there are A acres of grass,

- after one week with n oxen, the height of the grass is $(h + g - nd/A)$";
- after two weeks with n oxen, the height of the grass is $[(h + g - nd/A) + (g - nd/A)]$"

which is equal to $(h + 2g - 2nd/A)$" (n is a discrete variable)

- after x weeks with n oxen, the height of the grass is $(h + xg - xnd/A)$".

So, from the given data, with 1/3 replacing _,

$h + 4g - 4 \times 18 \times d/5 = 0$, that is to say

$$h + 4g - 72d/5 = 0 \qquad (1)$$

and $$h + 9g - 189d/10 = 0 \qquad (2)$$

(2) – (1) gives $5g = 9d/2$, and hence

$$g = 9d/10 \qquad (3)$$

From the last set of given data, we ask

If $$h + 18g - 18nd/24 = 0, \text{ what is } n? \qquad (4)$$

From (1), $h = 72d/5 - 4g$. Replace h by that expression in the equation in (4).

$$72d/5 + 14g - 3nd/4 = 0 \qquad (5)$$

From (3), $g = 9d/10$. Put that in (5).

$72d/5 + 63d/5 - 3nd/4 = 0$. Dividing each term by d gives

$3n/4 = 72/5 + 63d/5$

$3n/4 = 135/5 = 27$

which gives $n = 36$.

Concluding Comments

This paper has merely summarised what we know of the history of the pasturage problem. We do not yet know whether other writers added to the discussion surrounding the problem after Cajori's (1890) commentary. We were able to locate Robinson's award-winning solution (in Emerson, 1838) and have been pleased to include it in this paper. We were also pleased to be able to solve Emerson's version, and Newton's original version, using an algebraic approach. We intend to find the solution offered by the editor of the *Mathematical Monthly* in 1859, and also Dr Artemas Martin's solutions in 1882. Of course, we wish to locate a copy of Newton's own solution to his original problem.

One educational issue arising in relation to the problem was whether it is unwise to include very difficult problems in mathematics textbooks written for school students. Cajori (1890) criticised Emerson for including the problem. But the task was the last problem in a textbook specifically written for *advanced* scholars and, from that perspective, we think Emerson's decision to include it was educationally sound. The fact that the question could be linked to no less of an intellectual giant than Isaac Newton, made it all the more appropriate.

A second issue is whether mathematics teachers should be concerned to encourage "lucid" or "elegant" solutions to mathematics problems. Again, we would say "yes". We believe that for many non-trivial mathematics problems, some methods of finding solutions are "better" (in the sense that they are more "lucid", or more "elegant") than others, and it is a good idea to assist school students to appreciate that point. Having said that, it is not easy to specify criteria for ordering competing solutions for lucidity or elegance, and that was one of the problems the original committee faced when attempting to identify the most "lucid" solution of the 48 correct solutions submitted to the pasturage problem. It would be fascinating to be able to read the other 47 solutions but, probably, they are no longer extant.

Finally, we would be grateful if any readers of this paper who know anything more of the history of the pasturage problem would contact us. Also, we would be pleased if any person who devises a solution to the problem that is fundamentally different from our solution, and from Mr Robinson's, would send us the text of their solution. We hope to write a paper setting out, in greater detail than in this paper, the history of the pasturage problem, and we will gratefully acknowledge any assistance we receive from others.

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